PROBLEM SET 2

DUE: Mar.3

Problem 1

Show that the subgroup of index 2 must be normal.

Problem 2

Let m, n be integers, show that $\boldsymbol{Z}_m \times \boldsymbol{Z}_n \cong \boldsymbol{Z}_{mn}$, where $\boldsymbol{Z}_n$ is the group of integers modulo n.

Problem 3

Let G be a group. A **commutator** in G is an element of the form $aba^{-1}b^{-1}$ with $a, b \in G$. Let G^c be the subgroup generated by the commutators. The G^c is called the **commutator subgroup**. Show that G^c is normal. Show that any homomorphism of G into an abelian group factors through G/G^c.

Problem 4

Let H, K be subgroups of a finite group G with $K \subset N_H$. Show that

$$\sharp(HK) = \frac{\sharp(H)\sharp(K)}{\sharp(H \cap K)}$$

Problem 5 **Goursat's Lemma**

Let $G, G^{'}$ be groups, and let H be a subgroup of $G \times G^{'}$ such that the two projections $p_1 : H \to G$ and $p_2 : H \to G^{'}$ are surjective. Let N be the kernel of p_2 and $N^{'}$ be the kernel of p_1. One can identify N as a normal subgroup of G, and $N^{'}$ as a normal subgroup of $G^{'}$. Show that the image of H in $G/H \times G^{'}/H^{'}$ is the graph of an isomorphism

$$G/H \cong G^{'}/H^{'}$$

Problem 6

Prove that the group of inner automorphisms of a group G is normal in Aut(G).

Problem 7

Let G be a group and let H_1, H_2 be subgroups of finite index. Prove that $H_1 \cap H_2$ has finite index.

Semidirect product

Problem 8

Let G be a group and let H, N be subgroups with N normal. Let γ_x be conjugation by an element $x \in G$.

(a). Show that $x \mapsto \gamma_x$ induces a homomorphism $f : H \to Aut(N)$.

(b). If $H \cap N = \{e\}$, show that the map $H \times N \to HN$ given by $(x, y) \mapsto xy$ is a bijection, and that this map is an isomorphism if and only if f is trivial, i.e. $f(x) = id_N$ for all $x \in H$.

We define G to be the **semidirect product** of H and N if $G = NH$ and $H \cap N = \{e\}$.

(c). Conversely, let H, N be groups, and let $\psi : H \to Aut(N)$ be a given homomorphism. Construct a semidirect product as follows. Let G be the set of pairs (x, h) with $x \in N$ and $h \in H$. Define the composition law

$$(x_1, h_1)(x_2, h_2) = (x_1^{\psi(h_1)x_2}, h_1h_2)$$

Show that this is a group law, and yields a semidirect product of N and H, identifying N with the set of elements $(x, 1)$ and H the set of elements $(1, h)$.

Problem 9

(a). Let H, N be normal subgroups of a finite group G. Assume that the orders of H, N are relatively prime. Prove that $xy = yx$ for all $x \in H$ and $y \in N$, and that $H \times N \approx HN$.

(b). Let $H_1, ..., H_r$ be normal subgroups of G such that the order of H_i is relatively prime to the order of H_j for $i \neq j$. Prove that

$$H_1 \times .... \times H_r \approx H_1....H_r$$

.

Problem 10

Let G be a finite group and let N be a normal subgroup such that N and G/N have relatively prime orders.

(a). Let H be a subgroup of G having the same order as G/N. Prove that $G = HN$.

(b). Let g be an automorphism of G. Prove that $g(N) = N$.